Dynamic Replication and Hedging: A Reinforcement Learning Approach

Petter Kolm & Gordon Ritter
Courant Institute, NYU

petter.kolm@nyu.edu
https://www.linkedin.com/in/petterkolm

Published in Kolm and Ritter (2019), "Dynamic Replication and Hedging: A Reinforcement Learning Approach," *The Journal of Financial Data Science*

Applied Machine Learning Days
January 26-29, 2019

Outline

Machine learning in finance I

Adoption of ML has been rapid in the financial industry. Why?

- The data
- The algos & tools
- The infrastructure for data storage & computation

Difference between today versus 3 years ago

- Many use-cases are available now
- New technology constantly being adopted
  - Robo advisors, block chains
  - Many hedge funds use the Amazon S3 cloud, etc.
- Alternative data & its handling/preparation is better understood
  - A shift towards being incorporated into existing models
  - Common issues: Debiasing, tagging & paneling

Machine learning in finance II

Buyer beware:

- Testing models: In- versus out-of-sample
- Where are your algos and data from?
  - "Black box" risk on steroids
  - Is your data biased?
  - Are your algos correct? Do you know what they do?
- Data science & ML are *not* cure-alls (who knew?)

What do we want?

- Financial domain expertise often critical
- More business use cases

Machine learning for trading I

- The role of trading has not changed
- But modern ML allows us to solve more sophisticated models that have more features
  - Models are becoming more realistic
  - Models can be solved faster (yes, often sub-second)
  - Models can be made more adaptable to changing market conditions

Machine learning for trading II

ML tools that are being leveraged for trading:

- LASSO-based techniques yield very fast derivative free solvers for single- and multi-period problems
- Bayesian learning and Gaussian processes used for building forecasts (forecasting distributions) and dealing with estimation/model risk
- Unsupervised learning techniques for building statistical risk models
- Regime-switching models for dealing with different market environments (e.g. risk on/off)
- Mixed-distributions to model skewed distributions and tail-behavior
- Bootstrap and cross-validation techniques for backtesting
- Reinforcement learning (often tree- or NN-based models) for complex trading / planning problems in the presence of uncertainty (where the value function is not easily obtainable)

Replication & hedging

- Replicating and hedging an option position is fundamental in finance
- The core idea of the seminal work by Black-Scholes-Merton (BSM):
  - In a complete and frictionless market there is a continuously rebalanced dynamic trading strategy in the stock and riskless security that perfectly replicates the option (Black and Scholes (1973), Merton (1973))
- In practice continuous trading of arbitrarily small amounts of stock is infinitely costly and the replicating portfolio is adjusted at discrete times
  - Hence, perfect replication is impossible and an optimal hedging strategy will depend on the desired trade-off between replication error and trading costs

Related work I

While a number of articles have considered discrete time hedging or transaction costs alone,

- Leland (1985) was first to address discrete hedging under transaction costs
  - His work was subsequently followed by others[1]
  - The majority of these studies treat proportionate transaction costs
- More recently, several studies have considered option pricing and hedging subject to both permanent and temporary market impact in the spirit of Almgren and Chriss (1999), including Rogers and Singh (2010), Almgren and Li (2016), Bank, Soner, and Voß (2017), and Saito and Takahashi (2017)
- Halperin (2017) applies reinforcement learning to options but approach is specific to the BSM model and does not consider transaction costs

Related work II

- Buehler et al. (2018) evaluate NN-based hedging under convex risk measures subject to proportional transaction costs

[1] See, for example, Figlewski (1989), Boyle and Vorst (1992), Henrotte (1993), Grannan and Swindle (1996), Toft (1996), Whalley and Wilmott (1997), and Martellini (2000).

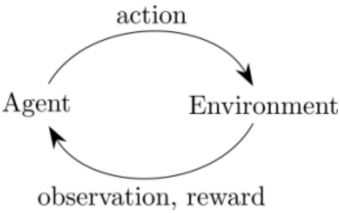
What we do

In the paper we:

- Show how to build a reinforcement learning (RL) system which can learn how to optimally hedge an option (or other derivative securities) in a fully realistic setting
    - Discrete time
    - Nonlinear transaction costs
    - Round-lotting
- Method allows the user to "plug-in" any option pricing and simulation library, and then train the system with no further modifications
- The system learns how to optimally trade-off trading costs versus hedging variance for that security
    - Uses a continuous state space
    - Relies on nonlinear regression techniques to the "sarsa targets" derived from the Bellman equation
- Method extends in a straightforward way to arbitrary portfolios of derivative securities

Brief introduction to RL I

- An interacting system: RL agent interacts with its environment. The "environment" is the part of the system outside of the agent's direct control
- At each time step t, the agent observes the current state of the environment s_t and chooses an action a_t from the action set
- This choice influences both the transition to the next state, as well as the reward the agent receives



Brief introduction to RL II

- A policy π is a way of choosing an action a_t, conditional on the current state s_t
- RL is the search for policies which maximize the expectation of the cumulative reward G_t

$$\mathbb{E}[G_t] = \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots]$$

  where γ is such that the infinite sum converges
- According to Sutton and Barto (2018) *"the key idea of RL generally, is the use of value functions to organize and structure the search for good policies"*
- Mathematically speaking, RL is a way to solve multi-period optimal control problems

Brief introduction to RL III

- The action-value function expresses the value of starting in state s, taking an arbitrary action a, and then following policy π thereafter

$$q_\pi(s, a) := \mathbb{E}_\pi[G_t \mid S_t = s, A_t = a] \tag{1}$$

  where $\mathbb{E}_\pi$ denotes the expectation under the assumption that policy π is followed
- If we knew the q-function corresponding to the optimal policy, say q, we would know the optimal policy itself

$$\text{choose } a \text{ in the action set to maximize } q(s_t, a)$$

  This is called following the *greedy policy*
- Hence the problem is reduced to finding q, or producing a sequence of iterates that converges to q
- Methods for producing those iterates are based on the Bellman equations

Automatic hedging in theory I

- We define automatic hedging to be the practice of using trained RL
- With no trading frictions and where continuous trading is possible, there may be a dynamic replicating portfolio which hedges the option position perfectly, meaning that the overall portfolio (option minus replication) has zero variance
- With frictions and where only discrete trading is possible the goal becomes to minimize variance and cost

Automatic hedging in theory II

- This suggest we can seek the agent's optimal portfolio as the solution to a mean-variance optimization problem with risk-aversion κ

$$\max\left(\mathbb{E}[w_T] - \frac{\kappa}{2}\mathbb{V}[w_T]\right) \tag{2}$$

where the final wealth w_T is the sum of individual wealth increments δw_t,

$$w_T = w_0 + \sum_{t=1}^{T} \delta w_t$$

Automatic hedging in theory III

- In the random walk case, this leads to solving

$$\min_{\text{permissible strategies}} \sum_{t=0}^{T} \left(\mathbb{E}[-\delta w_t] + \frac{\kappa}{2} \mathbb{V}[\delta w_t] \right) \tag{3}$$

  where

$$-\delta w_t = c_t$$

  and c_t is the total trading cost paid in period t (including commissions, bid-offer spread cost, market impact cost, and other sources of slippage)

- As shown in Ritter (2017), with an appropriate choice of the reward function the problem of maximizing this mean-variance problem can be recast as a RL problem. The reward in each period corresponding to (3) is approximately

$$R_t := \delta w_t - \frac{\kappa}{2} (\delta w_t)^2 \tag{4}$$

Automatic hedging in theory IV

- Thus, training reinforcement learners with this kind of reward function amounts to training automatic hedgers who tradeoff of costs versus hedging variance

Automatic hedging in practice I

- Simplest possible example: A European call option with strike price K and expiry T on a non-dividend-paying stock
- We take the strike and maturity as fixed, exogenously-given constants. For simplicity, we assume the risk-free rate is zero
- The agent we train will learn to hedge this specific option with this strike and maturity. It is not being trained to hedge any option with any possible strike/maturity

- For European options, the state must minimally contain (1) the current price S_t of the underlying, (2) the time $\tau := T - t > 0$ remaining to expiry, and (3) our current position of n shares
- The state is thus naturally an element of[2]

$$\mathcal{S} := \mathbb{R}_+^2 \times \mathbb{Z} = \{(S, \tau, n) \mid S > 0, \tau > 0, n \in \mathbb{Z}\}.$$

Automatic hedging in practice II

- The state *does not* need to contain the option Greeks, because they are (nonlinear) functions of the variables the agent has access to via the state
- We expect agents to learn such nonlinear functions on their own
- This has the advantage of not requiring any special, model-specific calculations that may not extend beyond BSM models

[2] If the option is American, then it may be optimal to exercise early just before an ex-dividend date. In this situation, the state must be augmented with one additional variable: The size of the anticipated dividend in period $t+1$.

Let's put RL at a disadvantage

- The RL agent is at a disadvantage: It does not know any of the following information:
  - the strike price K
  - that the stock price process is a geometric Brownian motion (GBM)
  - the volatility of the price process
  - the BSM formula
  - the payoff function $(S - K)_+$ at maturity
  - any of the Greeks

  Thus, it must infer the relevant information, insofar as it affects the value function, by interacting with a simulated environment

Simulation assumptions I

- We simulate a discrete BSM world where the stock price process is a geometric Brownian motion (GBM) with initial price S_0 and daily lognormal volatility of σ/day
- We consider an initially at-the-money European call option (struck at $K = S_0$) with T days to maturity
- We discretize time with D periods per day, hence each "episode" has $T \cdot D$ total periods
- We require trades (hence also holdings) to be integer numbers of shares
- We assume that our agent's job is to hedge one contract of this option
- In the specific examples below, the parameters are $\sigma = 0.01, S_0 = 100, T = 10$, and $D = 5$. We set the risk-aversion, $\kappa = 0.1$

Simulation assumptions II

- T-costs: For a trade size of n shares we define

$$\text{cost}(n) = \text{multiplier} \times \text{TickSize} \times (|n| + 0.01n^2)$$

  where we take $\text{TickSize} = 0.1$. With $\text{multiplier} = 1$, the term $\text{TickSize} \times |n|$ represents a cost, relative to the midpoint, of crossing a bid-offer spread that is two ticks wide. The quadratic term is a simplistic model for market impact
- All of the examples were trained on a single CPU, and the longest training time allowed was one hour

Example: Baseline agent (discrete & no t-costs)

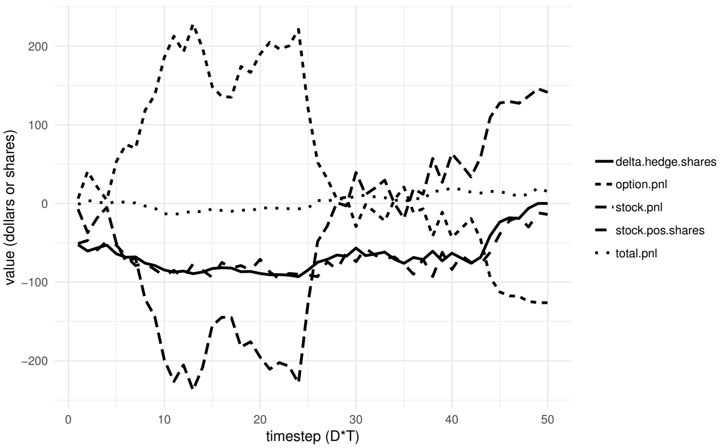


Figure 1: Stock & options P&L roughly cancel to give the (relatively low variance) total P&L. The agent's position tracks the delta

Example: Baseline agent (discrete & t-costs)

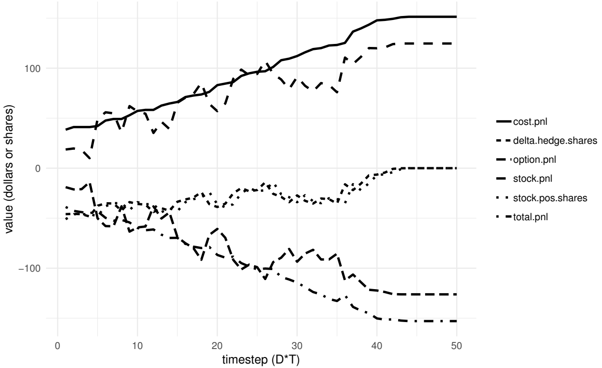
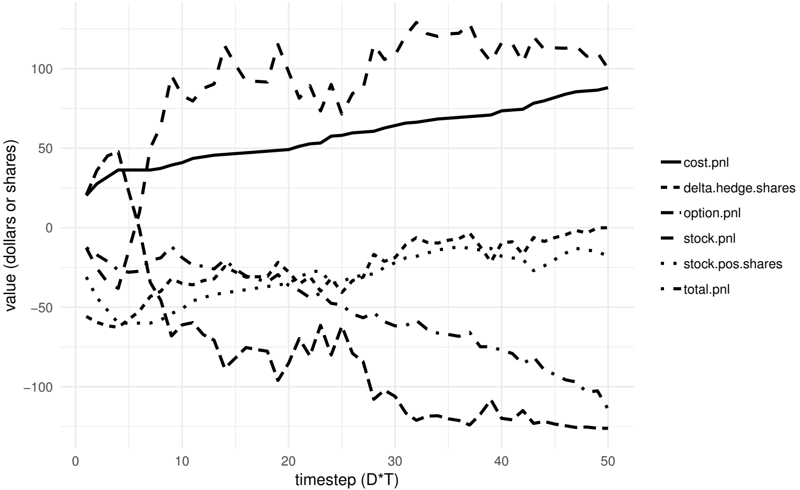


Figure 2: Stock & options P&L roughly cancel to give the (relatively low variance) total P&L. The agent trades so that the position in the next period will be the quantity $-100 \cdot \Delta$ rounded to shares

Example: T-cost aware agent (discrete & t-costs)



Kernel density estimates of total cost & volatility

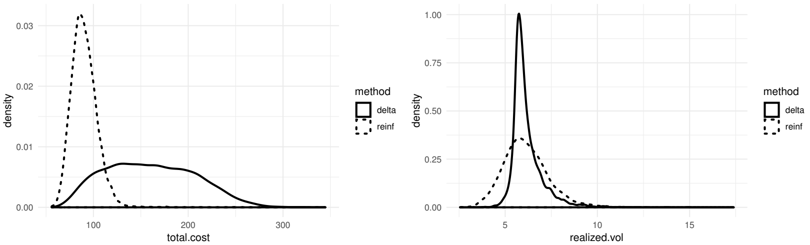


Figure 3: Kernel density estimates for total cost (left panel) and volatility of total P&L (right panel) from $N = 10,000$ out-of-sample simulations. The "reinf" policy achieves much lower cost (t-statistic $= -143.22$) with no significant difference in volatility of total P&L.

Kernel density estimates of total P&L

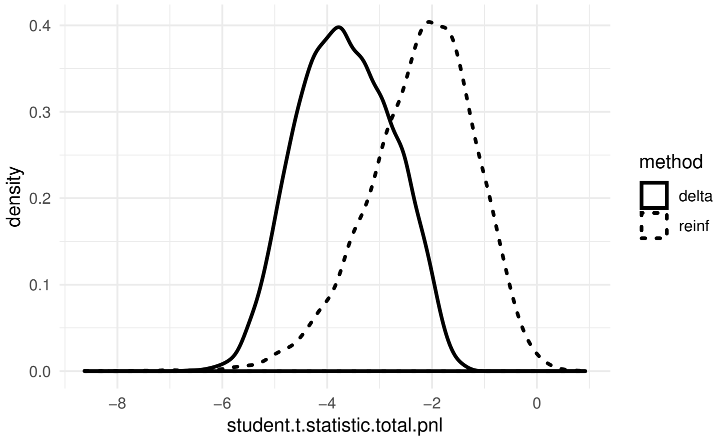


Figure 4: Kernel density estimates of the t-statistic of total P&L for each of our out-of-sample simulation runs, and for both policies represented above ("delta" and "reinf"). The "reinf" method is seen to outperform in the sense that the t-statistic is much more often close to zero and insignificant.

Conclusions I

We have introduced a reinforcement learning system that hedges an option under realistic conditions of discrete trading and t-costs

- The approach does not depend on the existence of perfect dynamic replication. The system learns to optimally trade off variance and cost, as best as possible using whatever securities it is given as potential candidates for inclusion in the replicating portfolio
- It accomplishes this without the user providing any of the following information:
  - the strike price K
  - the fact that the stock price process
  - the volatility of the price process
  - the Black-Scholes-Merton (BSM) formula
  - the payoff function $(S - K)_+$ at maturity
  - any of the Greeks

Conclusions II

- A key strength of the RL approach: It does not make any assumptions about the form of t-costs. RL learns the minimum variance hedge subject to whatever t-cost function one provides. All it needs is a good simulator, in which t-costs and options prices are simulated accurately

References I

Almgren, Robert and Neil Chriss (1999). "Value under liquidation". In: *Risk* 12.12, pp. 61–63.

Almgren, Robert and Tianhui Michael Li (2016). "Option hedging with smooth market impact". In: *Market Microstructure and Liquidity* 2.1, p. 1650002.

Bank, Peter, H Mete Soner, and Moritz Voß (2017). "Hedging with temporary price impact". In: *Mathematics and Financial Economics* 11.2, pp. 215–239.

Black, Fischer and Myron Scholes (1973). "The pricing of options and corporate liabilities". In: *Journal of Political Economy* 81.3, pp. 637–654.

Boyle, Phelim P and Ton Vorst (1992). "Option replication in discrete time with transaction costs". In: *The Journal of Finance* 47.1, pp. 271–293.

Buehler, Hans et al. (2018). "Deep hedging". In: *arXiv:1802.03042*.

Figlewski, Stephen (1989). "Options arbitrage in imperfect markets". In: *The Journal of Finance* 44.5, pp. 1289–1311.

Grannan, Erik R and Glen H Swindle (1996). "Minimizing transaction costs of option hedging strategies". In: *Mathematical Finance* 6.4, pp. 341–364.

Halperin, Igor (2017). "QLBS: Q-Learner in the Black-Scholes (-Merton) Worlds". In: *arXiv:1712.04609*.

Henrotte, Philippe (1993). "Transaction costs and duplication strategies". In: *Graduate School of Business, Stanford University*.

References II

Kolm, Petter and Gordon Ritter (2019). "Dynamic Replication and Hedging: A Reinforcement Learning Approach". In: *The Journal of Financial Data Science* 1.1.

Leland, Hayne E (1985). "Option pricing and replication with transactions costs". In: *The Journal of Finance* 40.5, pp. 1283–1301.

Martellini, Lionel (2000). "Efficient option replication in the presence of transactions costs". In: *Review of Derivatives Research* 4.2, pp. 107–131.

Merton, Robert C (1973). "Theory of rational option pricing". In: *The Bell Journal of Economics and Management Science*, pp. 141–183.

Ritter, Gordon (2017). "Machine Learning for Trading". In: *Risk* 30.10, pp. 84–89.

Rogers, Leonard CG and Surbjeet Singh (2010). "The cost of illiquidity and its effects on hedging". In: *Mathematical Finance* 20.4, pp. 597–615.

Saito, Taiga and Akihiko Takahashi (2017). "Derivatives pricing with market impact and limit order book". In: *Automatica* 86, pp. 154–165.

Sutton, Richard S and Andrew G Barto (2018). *Reinforcement learning: An introduction*. Second edition, in progress. MIT press Cambridge.

Toft, Klaus Bjerre (1996). "On the mean-variance tradeoff in option replication with transactions costs". In: *Journal of Financial and Quantitative Analysis* 31.2, pp. 233–263.

Whalley, A Elizabeth and Paul Wilmott (1997). "An asymptotic analysis of an optimal hedging model for option pricing with transaction costs". In: *Mathematical Finance* 7.3, pp. 307–324.