# Deep optimal stopping

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The decision to stop at time t must be based on  $X_0, \ldots, X_t$ !



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Existing methods suffer from the curse of dimensionality and are therefore not feasible for large d!

# Deep learning approach

- Learn a candidate optimal stopping time  $\hat{\tau} : \Omega \to \{0, \tau/n, ..., T\}$ , i.e., for every  $t \in \{0, \tau/n, ..., T\}$  train a neural network  $f_t : \mathbb{R}^d \to \{0, 1\}$  that decides to stop or not
  - $L = \mathbb{E}[g(\hat{ au}, X_{\hat{ au}})]$  is a lower bound for  $\sup_{ au} \mathbb{E}[g( au, X_{ au})]$
  - **Calculate a Monte Carlo estimate**  $\hat{L} = \frac{1}{M} \sum_{m=1}^{M} L_m$  for L



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Consider d assets in a multi-dimensional Black-Scholes model

$$X_t^i = s_0^i \exp\left([r - \delta_i - \sigma_i^2/2]t + \sigma_i W_t^i\right), \quad i \in \{1, 2, \dots, d\},$$

where  $\mathbf{s}_0^i \in (0, \infty)$  (initial values),  $r \in \mathbb{R}$  (risk-free interest rate),  $\delta_l \in [0, \infty)$ (dividend yields),  $\sigma_l \in (0, \infty)$  (volatilities), and  $(W_l)_{l \in [0, T]}$  is a *d*-dimensional Wiener process.

A Bermudan max-call option has time-t payoff (max<sub>1 $\leq i \leq d$ </sub> X<sup>1</sup><sub>t</sub> - and can be exercised at one of finitely many times

$$= t_0 < t_1 = \frac{T}{N} < t_2 = 3\frac{2T}{N} < \dots < t_N = T.$$
Value: 
$$\sum_{\substack{\tau \in \{t_0, t_1, \dots, T\}}}^{l} \mathbb{E} \left[ e^{-r\tau} \left( \max_{\substack{1 \le t \le d}} X_{\tau}^t - K \right) \right]$$

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d	L	tL	U	tυ	Point est.	95% CI	Binomial	BC 95 $\%$ CI
2	13.901	27.6	113.903	4.1	13.902	[13.892, 13.932]	13.902	



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5	26.145	30.2	26.165	4.3	26.155	[26.126, 26.203]		[26.115, 26.164]



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	13.902	[13.892, 13.932]	13.902	4.1	13.903	27.6	13.901	2
	18.69	[18.677, 18.744]	18.702	4.1	18.710	27.9	18.694	3
[26.115, 26.164]		[26.126, 26.203]	26.155	4.3	26.165	30.2	26.145	5
		[38.332, 38.401]	38.355	4.6	38.357	32.2	38.353	10
		[51.562, 51.856]	51.690	5.5	51.796	37.4	51.584	20
		[59.490, 59.869]	59.657	6.1	59.802	43.4	59.512	30
		[69.559, 70.101]	69.795	7.5	70.008	55.6	69.582	50
		[83.355, 83.860]	83.579	11.7	83.779	90.5	83.378	100
		[97.398, 97.851]	97.594	19.3	97.767	161.2	97.422	200
		[116.239, 116.733]	116.455	48.1	116.645	450.5	116.264	500



# ZENA

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